Gila Hanna
Hans Niels Jahnke
Helmut Pulte
Editors

Explanation and Proof in Mathematics

Philosophical and Educational Perspectives

Springer

Gila Hanna · Hans Niels Jahnke
Helmut Pulte
Editors

Explanation and Proof in Mathematics

Philosophical and Educational Perspectives

Springer

Editors
Gila Hanna
Ontario Institute for Studies
in Education (OISE)
University of Toronto
Toronto ON, Canada
ghanna@oise.utoronto.ca

Hans Niels Jahnke
Department of Mathematics
University of Duisburg-Essen
Essen
Germany
njahnke@uni-due.de

Helmut Pulte
Ruhr-Universität
Bochum
Germany
helmut.pulte@rub.de

ISBN 978-1-4419-0575-8 e-ISBN 978-1-4419-0576-5
DOI 10.1007/978-1-4419-0576-5
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2009933111

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Chapter 1
Introduction

The essays collected in this volume were originally contributions to the conference *Explanation and Proof in Mathematics: Philosophical and Educational Perspectives*, which was held in Essen in November 2006. The essays are substantially extended versions of those papers presented at the conference; each essay has been reviewed by two reviewers and has undergone criticism and revision.

The conference was organized by the editors of this volume and brought together people from the fields of mathematics education, philosophy of mathematics and history of mathematics. The conference organizers firmly believe that this interdisciplinary dialog on proof between scholars in these three fields will be fruitful and rewarding for each field for several reasons.

Developments in the *practice of mathematics* during the last 3 decades have led to new types of proof and argumentation, challenging the established norms in this area. These developments originated from the use of computers (both as heuristic devices and as means of verification), from a new quality in the relations of mathematics to its applications in the empirical sciences and technology (see the Jaffe–Quinn paper and the subsequent debate among mathematicians, for example), and from a stronger awareness of the social nature of the processes leading to the acceptance of a proof.

These developments reflect the philosophy of mathematics, partly *ex post facto*, and partly in *anticipation*. Philosophers have long sought to define the nature of mathematics, notably by focusing upon its *logical foundations* and its formal structure. Over the past 40 years, however, the focus has shifted to encompass epistemological issues such as visualization, explanation and diagrammatic thinking.

As a consequence, in the *philosophy and history of mathematics* the approach to understanding mathematics has changed dramatically. More attention is paid to mathematical practice. This change was first highlighted in the late 1960s by the work of Imre Lakatos, who pronounced mathematics a "quasi-empirical science." His work continues to be highly relevant for the philosophy of mathematics as well as for the educational aspects of mathematics.

The work of Lakatos and others gave rise to conceptions of mathematics in general, and of proof in particular, based on detailed studies of mathematical practice. Recently, these studies have been frequently combined with the epistemological points

G. Hanna et al. (eds.), *Explanation and Proof in Mathematics: Philosophical and Educational Perspectives*, DOI 10.1007/978-1-4419-0576-5_1,

of view and cognitive approaches commonly subsumed under the term "naturalism." In this context, philosophers have come to a greater recognition of the central importance of mathematical understanding, and so have looked more closely at how understanding is conveyed and at what counts as explanation in mathematics. As might be expected given these two changes in focus, philosophers of mathematics have turned their attention more and more from the *justificatory* to the *explanatory* role of proof. Their central questions are no longer only why and how a proof makes a proposition true but also how it contributes to an adequate understanding of the proposition and what role is played in this process by factors that go beyond logic.

The computer has caused a radical change in *educational practices* as well. In algebra, analysis, geometry and statistics, for example, computer software already provides revolutionary capabilities for visualization and *experimentation*, and holds the promise of still more change. In sum, trends in the philosophy and history of mathematics, as well as in mathematics education, have lead to a diversity of notions of proof and explanation. These trends interact, as people in one field are sensitive to developments in the others. The tendencies in the different fields are not identical, however; each field retains its own peculiarities.

The present volume intends to strengthen, in particular, mutual awareness in the philosophy of mathematics and in mathematics education about these new developments and to contribute to a more coherent theoretical framework based upon recent advances in the different fields. This seems quite possible (even necessary) in light of the strong empirical and realistic tendencies now shared by philosophy of mathematics and mathematics education. More important, though they share a strong interest in these new understandings of mathematical explanation and proof, philosophers of mathematics and researchers in mathematics education usually work in different institutional settings and in different research programs. It is crucial that researchers in both fields take an interest in the problems and questions of the other. So, we invited philosophers and historians to reflect on which dimensions of mathematical proof and explanation could be relevant to the general culture and to broadly educated adults and asked people from didactics to specifically elicit the epistemological and methodological aspects of their ideas.

In preparing the conference we identified four subthemes to help organize this dialog between philosophers of mathematics and mathematics educators. They refer to central concerns of the two groups as well as designating issues on which both groups are currently working:

1.1 The Legacy of Lakatos

Lakatos' conception of mathematics as a "quasi-empirical science" has proved influential for the philosophy of mathematics as well as for the educational context. Though the naïve idea that Lakatos' concepts could be transferred directly into the classroom, in the hope that insights into the need for proof would arise immediately from classroom discussions, has been proven untenable, Lakatos' work is still an inspiration for both philosophers and educators.

1.2 Diagrammatic Thinking

The term "diagrammatic thinking" was coined by C. S. Peirce to designate the fact that thinking cannot be explained by purely logical means but is deeply dependent upon the systems of symbols and representations that are used. Independently of Peirce and philosophical discourse, this idea plays a key role in the didactics of mathematics, particularly in relation to mathematical argumentation and proof.

1.3 Mathematical Proof and the Empirical Sciences

A number of authors conceive of *mathematics in its connection with the empirical sciences*, especially physics. One can designate this approach as a form of physicalism – albeit in the broad meaning of that term. This does not at all mean that mathematics itself is considered to be an empirical science in a strict epistemological sense. This position stresses, rather, that the contents, methods and meaning of mathematics are to be discussed under the point of view that mathematics contributes, via the empirical sciences, to our understanding of the world around us. Theoretical concepts of mathematics, such as group and vector space, are to be set on a par with theoretical concepts of physics, such as electron and electromagnetic wave.

1.4 Different Types of Reasoning and Proof

In the practice and teaching of mathematics, different forms of mathematical argumentation have evolved; some of these are considered as proofs proper and some as heuristic devices. Besides formal proofs, we mention the various forms of induction, analogy, enumeration, algebraic manipulation, visualization, computer experimentation, computer proof and modeling. The conference tried to understand these modes of argumentation better and in greater depth, and to analyse the different views of their acceptability and fruitfulness on the part of mathematicians, philosophers and mathematics educators.

As it turned out, the subthemes proved to be recurring issues which surfaced in various papers rather than suitable bases for grouping them. Hence, we decided to organize the essays for the book in three broad sections.

Part I, "*Reflections on the nature and teaching of proof,*" has seven papers belonging to the first, third and fourth main themes of the conference. Lakatos' philosophy of mathematics is discussed or applied mainly in Koetsier's and Larvor's articles. The function of proof and explanation in mathematics and the empirical sciences plays a more or less prominent role in Jahnke's and Mormann's papers. Different theoretical types of proofs and their practical implications are central to the papers of Leng and Hanna and Barbeau. Heinze's paper plays a special role, because it deals with mathematical proofs neither from the point of view of the philosopher or historian of mathematics nor from that of mathematical educators, but brings in the perspective of working mathematicians.

Hans Niels Jahnke's paper *"The Conjoint Origin of Proof and Theoretical Physics"* "triangulates" the historical, philosophical and educational aspects of the idea of mathematical proof in ancient Greece. Jahnke argues that the rise of mathematical proof cannot be understood solely as an outcome of social-political processes or of internal mathematical developments, but rather as the result of a fruitful interaction of both. Following mainly A. Szabó's path-breaking historical studies of the concept of proof, Jahnke argues that mathematical proof – at least in the early context of dialectic – was understood as a mode of rational discourse not restricted to the aim of securing "dogmatic" claims. It mainly served to defend plausible presuppositions and to organize mathematical knowledge in an axiomatic-deductive manner. Setting up axioms and deducing theorems therefore were by no means unique to mathematics proper (i.e., geometry and arithmetic) but were also applied in other fields of knowledge, especially in those areas later considered parts of theoretical physics (e.g., statics, hydrostatics, astronomy). Jahnke then integrates into his argument P. Maddy's distinction between "intrinsic" and "extrinsic" justification of axioms in order to show that (pure) mathematics in the twentieth century could not have evolved without an extrinsic motivation and justification of basic hypotheses of mathematics and therefore shows marked similarities to the early Greek tradition. Consequently, Jahnke argues for a "new" manner of discussing mathematical proof in the classroom not only by integrating elements of the "old" dialectical tradition, but also by rejecting excessive, outmoded epistemological claims about mathematical axioms and proofs.

Teun Koetsier's contribution *"Lakatos, Lakoff and Núnez: Towards a Satisfactory Definition of Continuity"* aims to integrate Lakatos' logic and methodology of mathematics, as highlighted in his famous *Proofs and Refutations*, and Lakoff and Núnez's theory of metaphorical thinking in mathematics. To do this, Koetsier introduces the evolution of the concept of continuity from Euclid to the late nineteenth century as a case study. He argues that this development can be understood as a successive transformation of conceptual metaphors which starts from the "Euclidean Metaphor" of geometry and ends (via Leibniz, Euler, Lagrange, Encontre, Cauchy, Heine and Dedekind) in a quite modern, though seemingly (also) *metaphorical* treatment of the intermediate-value principle of analysis. In this case study, Koetsier conventionally presents mathematics as a system of conceptual metaphors in Lakoff & Núnez's sense. At the same time, he proposes a Lakatosian interplay of analysis and synthesis as a motor of system-transformations and as a warrantor of mathematical progress: Conjectures are turned into propositions and are (later) rejected by means of analysis and synthesis. The subsequent application of these methods leads to a continued elaboration and refinement of mathematical concepts (metaphors?) and techniques.

Mary Leng's paper *"Pre-Axiomatic Mathematical Reasoning: An Algebraic Approach"* takes a different position with respect to mathematical proof and mathematical theorizing in general. Following G. Hellman's terminology, Leng introduces an "algebraic approach" in a partly metaphorical manner in order to characterize the view that axioms relate to mathematical objects analogously to how algebraic equations with unknown variables relate to their solutions (which may form different, varied systems). Leng contrasts this approach, which comes close to Hilbert's, with the "assertory" approach of Frege and others, which holds that axioms are assertions of truths about a particular set of objects given independently of the axioms. Leng gives an account of the pros and cons of both views with respect to the truth of axioms in general and to the reference of mathematical propositions. She pays special attention to the fact that a lot of "pre-axiomatised" mathematics is done: namely, mathematics that apparently refers to well-established mathematical objects not "given" by formal axioms. Leng defends a "liberal" algebraic view which can deal with pre-axiomatic mathematical theorizing without getting caught in the traps of traditional "algebraic" and "assertory" approaches to axiomatisation.

Thomas Mormann's *"Completions, Constructions and Corrollaries"* brings a "Kantian" perspective to mathematical proof and to the general formation and development of mathematical concepts. Mormann focuses on Cassirer's theory of idealization in relation to Kant's theory of intuition as well as to Peirce's so-called "theorematic reasoning." First, he outlines Kant's understanding of intuition in mathematics and its main function – controlling mathematical proofs by constructive step-by-step checks. Then, he presents Russell's logicism as the "anti-intuitive" opponent of the Kantian philosophy of mathematics. Despite this antagonism, Mormann posits that both positions argued for a fixed, stable framework for mathematics, rooted in intuition or relational logic respectively. In his reconstruction, Mormann considers Cassirer's "critical idealism" as a sublime synthesis of both precursors, which eliminates the sharp philosophical separation between mathematics and the empirical sciences: Cassirer's concept of idealization is an "overarching" principle, being effective in both mathematics and the empirical sciences. Further, Mormann argues, this procedure of idealization is basic for some "completions" in mathematics (like Hilbert's principle of continuity) which are not secured by a purely logical approach. Mormann presents Peirce's "theorematic reasoning" as a kind of complement in order to make Cassirer's completions work in mathematical practice. The "common denominator" of both approaches, according to Mormann, is a shift in the general understanding of philosophy of mathematics: Its main task is no longer to provide unshakable foundations for mathematics and science but to analyze the formation and transformation of general concepts and their functions in mathematical and scientific practice.

Brandon Larvor's contribution *"Authoritarian vs. Authoritative Teaching: Polya and Lakatos"* endeavors to understand and compare the two mathematicians' theories of mathematical education arising from their (common) "Hungarian" mathematical tradition that started with L. Fejer. Larvor shows that Lakatos' "critical" and "heuristic" approach to teaching, which later culminates in his *Proofs and Refutations*, is already present in his early statements on the role of education and science and might have been shaped by his mathematical teacher Sándor Karácsony. Lakatos' "egalitarian" understanding of teaching mathematics is rooted in a political distaste for authoritarianism. His appreciation of heuristic proofs at the expense of deductive proofs is perhaps the most visible result of this distaste. According to Larvor, however, Lakatos failed to develop a useful pedagogical model that takes into account the basic fact that students and teachers are not equal dialog partners. Polya, on the other hand, stressed earlier than Lakatos the distinction between deductive and heuristic presentations of mathematics and made explicit the "shaping" function of heuristics in mathematical proof. Contrary to Lakatos, he develops a model of teaching mathematics; his model is not egalitarian, but aims at a kind of "mathematical empathy" in the relation of experienced teacher and learning student. Polya also rejects mathematical fallibilism, which is important for Lakatos' philosophy of mathematics. Though both thinkers share important insights into the teaching of mathematics, Lakatos' understanding might be described as anti-authoritative, while Polya's can be described as "authoritative," though not as "authoritarian."

Gila Hanna's and Ed Barbeau's paper *"Proofs as Bearers of Mathematical Knowledge"* extends Yehuda Rav's thesis that mathematical proofs (rather than theorems) should be the main focus of mathematical interest: They are the primary bearers of mathematical knowledge, if this knowledge is not restricted to results and their truth but is understood as the ability to apply methods, tools, strategies and concepts. In the first part of the paper, Hanna and Barbeau present and analyze Rav's thesis and its further development in its original context of mathematical practice. Here, informal proofs – "conceptual proofs" instead of formal derivations – dominate mathematical argumentation and are of special importance. Among other arguments, Rav's thesis gains considerable support from the fact that mathematical theorems often are re-proven differently (J. W. Dawson), even if their "truth-preserving" function is beyond doubt. The second part of the paper aims at a *desideratum* of mathematical education in applying and transforming Rav's concept of proof to teaching mathematics. With special reference to detailed analysis of two case studies from algebra and geometry, Hanna and Barbeau argue that conceptual proofs deserve a major role in advanced mathematical education, because they are of primary importance for the teaching of methods and strategies. This kind of teaching proofs is not meant as a challenge to "Euclidean" proofs in the classroom but as a complement which broadens the view of mathematical proof and the nature of mathematics in general.

Aiso Heinze's *"Mathematicians' Individual Criteria for Accepting Theorems and Proofs: An Empirical Approach"* enlarges and concludes the "Reflections" of Part I through an empirical study on the working mathematician's views on proof. When is the mathematical community prepared to accept a proposed proof as such? The social processes and criteria of evaluation involved in answering this question are at the core of Heinze's explorative (though not representative) empirical investigation, which surveyed 40 mathematicians from southern Germany. The survey questions referred to a couple of possible criteria for the individual acceptance of proofs which belong to the participants' own research areas, to other research areas or to part of a research article which has to be reviewed. Some of the findings are hardly astonishing – a trust in peer-review processes and in the judgment of the larger mathematical community – but also the personal checking of a proof in some detail plays a major role. Particularly, senior mathematicians frequently do not automatically accept "second-hand" checks as correct. Apparently, a skeptical and individualistic attitude within the mathematical community goes hand in hand with the epistemological fact that a deeper understanding of proven theorems needs a reconstruction of the proof-process itself. Due to the lack of further empirical data, however, these and other conjectures are open to further discussion and investigation.

Part II of the book, "*Proof and cognitive development,*" consists of four papers. The first two investigate promising theoretical frameworks, whereas the last two use a well-established Vygotskian framework to examine results of empirical research.

In "*Bridging Knowing and Proving in Mathematics: A Didactical Perspective,*" Nicolas Balacheff begins by identifying two didactical gaps that confront new secondary school students. First, they have not yet learned that proof in mathematics is very different from what counts as evidence in other disciplines, including the physical sciences. Second, they have studied mathematics for years without being told about mathematical proof, but as soon as they get to secondary school they are abruptly introduced to proof as an essential part of mathematics and find themselves having to cope with understanding and constructing mathematical proofs.

These gaps make the teaching of mathematics difficult; in Balacheff's view, they point to the need to examine the teaching and learning of mathematical proof as a "mastery of the relationships among knowing, representing and proving mathematically." The bulk of his paper is devoted to developing a framework for understanding the didactical complexity of learning and teaching mathematical proof, in particular for analyzing the gap between knowing mathematics and proving in mathematics.

Seeking such a framework, Balacheff characterizes the relationship between proof and explanation quite differently from most contemporary philosophers of mathematics, who discuss the explanatory power of proofs on the premise that not all mathematical proofs explain and not all mathematical explanations are proofs. Balacheff, however, states that a proof is an explanation by virtue of being a proof.

He sees a proof as starting out as a text (a candidate-proof) that goes through three stages. In the first stage, the text is meant to be an explanation. In the second stage, this text (explanation) undergoes a process of validation (an appropriate community regards that text as a proof). Finally, in the third stage the text (now considered a proof by the appropriate community) is judged to meet the current standards of mathematical practice and thus becomes a legitimate mathematical proof. As Balacheff's Venn diagram shows, a proof is embedded in the class of explanation, that is, "mathematical proof $\subseteq$ proof $\subseteq$ explanation." Balacheff then arrives at a framework with three components: (1) action, (2) formulation (semiotic system), and (3) validation (control structure). He concludes that "This trilogy, which defines a conception, also shapes didactical situations; there is no validation possible if a claim has not been explicitly expressed and shared; and there is no representation without a semantic which emerges from the activity (i.e., from the interaction of the learner with the mathematical milieu)."

In *"The Long-term Cognitive Development of Reasoning and Proof,"* David Tall and Juan Pablo Mejia-Ramos use Tall's model of "three worlds of mathematics" to discuss aspects of cognitive development in mathematical thinking. In his previous research, Tall investigated for more than 30 years how children come to understand mathematics. His results, published in several scholarly journals, led him to define "three worlds of mathematics" – three ways in which individuals operate when faced with new learning tasks: (1) conceptual-embodied (using physical, visual and other senses); (2) proceptual-symbolic (using mathematical symbols as both processes and concepts, thus the term "procept"), and (3) axiomatic-formal (using formal mathematics).

Tall and Mejia-Ramos examine the difficult transition experienced by university students, from somewhat informal reasoning in school mathematics to proving within the formal theory of mathematics. Using Tall's "three worlds" model in combination with Toulmin's theory of argumentation, they describe how the three worlds overlap to a certain degree and are also interdependent. The first two worlds, those of embodiment and symbolism, do act as a foundation for progress towards the axiomatic-formal world. But the third, axiomatic-formal, world also acts as a foundation for the first two worlds, in that it often leads back to new and different worlds of embodiment and symbolism.

Tall and Mejia-Ramos argue that an understanding of formalization is insufficient to understand proof, since they have shown "how not only does embodiment and symbolism lead into formal proof, but how structure theorems return us to more powerful forms of embodiment and symbolism that can support the quest for further development of ideas."

The next two papers, "*Historical Artefacts, Semiotic Mediation, and Teaching Proof*" by Mariolina Bartolini-Bussi, and *"Proofs, Semiotics and Artefacts of Information Technologies"* by Alessandra Mariotti, also investigate the cognitive challenges in teaching and learning proof, but they do not aim at analyzing existing theoretical frameworks or developing new ones. Rather, they both use Vygotsky's framework, the basic assumptions of which are that the individual mind is an active participant in cognition and that learning is an essentially social process with a semiotic character, requiring interpretation and reconstruction of communication signs and artefacts. A key point of Vygotsky's theory is the need for mediation between the individual mind and the external social world. Bartolini-Bussi and Mariotti both explore the use of artefacts in the mathematics classroom and try to understand how these artefacts act as a means of mediation and how their use enables students to make sense of new learning tasks.

Bartolini-Bussi examines concrete physical artefacts: a pair of gear wheels meshed so that turning one causes the other to turn in the opposite direction, and mechanical devices for constructing parabolas. In the case of the gear wheels, the use of a concrete artefact proved to be helpful, in that students did come up with a postulate and a conviction that their postulate would be validated. In addition, the use the artefact seemed to have fostered a semiotic activity that encouraged the students to reason more theoretically about the functioning of gears.

In the case of the mechanical devices for constructing parabolas, Bartolini-Bussi notes that these concrete artefacts offered several advantages: (1) a context for historical reconstruction, for dynamic exploration and for the production of a conjecture, (2) continuous support during the construction of a proof framed by elementary geometry, and (3) a demonstration of the geometrical meaning of the parameter "p" that appears in the conic equation.

Mariotti examines two information technology artefacts: *Cabri-géomètre*, a dynamic geometry program, and *L'Algebrista*, a symbolic manipulation program. She uses the semiotic character of these specific artefacts to help students approach issues of validation and to teach mathematical proof. Mariotti gives an example of how the use of the Dynamic Geometry Environment artefact, *Cabri-géomètre*, carries semiotic potential and thus is useful a tool in teaching proof. This artefact enabled the teachers to structure classroom activities whereby students were engaged in (1) the production of a *Cabri* figure corresponding to a geometric figure, (2) a description of the procedure used to obtain the *Cabri* figure, and (3) a justification of the "correctness" of the construction. A second example, concerning the teaching of algebraic theory, uses a symbolic manipulator, *L'Algebrista*, as an artefact. Again, this artefact allowed a restructuring of classroom activities that enabled teachers to increase mathematical meanings for their students.

These two papers lend support to the idea that semiotic mediators in the form of artefacts, whether physical or derived from information technology, can be used successfully in the classroom at both the elementary and the secondary levels, not only to teach mathematics but to help students understand how one arrives at mathematical validation.

Part III, "*Experiments, Diagrams and Proofs*," analyzes the phenomenon of proof by considering the interaction between processes and products. The first essay in this part, by a philosopher of mathematics, sets the stage with a fresh view of Wittgenstein's ideas on proof. Two essays on educational issues follow, which put proof in the broader context of experimentation and problem solving. Part III is completed by two historical case studies relating the process of proving to the way a proof is written down.

In "*Proof as Experiment in Wittgenstein*," Alfred Nordmann reconstructs Wittgenstein's philosophy of mathematical proof as a complementarity between "proof as picture" and "proof as experiment." The perspectives designated by these two concepts are quite different; consequently, philosophers have produced bewilderingly different interpretations of Wittgenstein's approach. Using the concept of "complementarity," Nordmann invites the reader to consider these two perspectives as necessarily related, thus reconciling the seemingly divergent interpretations. However, he leaves open the question of whether every proof can be considered in both these ways.

"Proof as picture" refers to a proof as a written product. For Wittgenstein, it is exemplified by a *calculation* as it appears on a sheet of paper. Such a calculation comprises, line-by-line, the steps which lead from the initial assumptions to the final result. It shows two features: It is (1) surveyable and (2) reproducible. On the one hand, only the proof as a surveyable whole can tell us what was proved.

On the other hand, the proof can also be reproduced "with certainty in its entirety" like copying a picture wholesale and "once and for all."

"Proof as experiment" relates to the productive and creative aspects of proof. In an analogy to scientific experiments, the term refers to the experience of undergoing the proof. Wittgenstein's paradigm case for this view is the *reductio ad absurdum* or negative proof. In this case, a proof does not add a conclusion to the premises but it changes the domain of what is imaginable by rejecting one of the premises. Hence, going through the proof involves us in a process at the end of which we see things differently. For example, proving that trisection of an angle by ruler and compass is impossible changes our idea of trisection itself. The proof allows us to shift from an old to a new state, from a wrong way of seeing the world to a right one.

Nordmann argues that the opposition between pictures and experiments elucidates what is vaguely designated by opposing static vs. dynamic, synchronic vs. diachronic, and justificatory vs. exploratory aspects of proof. Proof as picture and proof as experiment are two ways of considering proof rather than two types of proof. They cannot be distinguished as necessary on the one hand and empirical on the other. The experiments of the mathematician and of the empirical scientist are similar in that neither experimenter knows what the results will be, but differ in that the mathematicians' experiment immediately yields a surveyable picture of itself, so that showing something and showing its paradigmatic necessity can collapse into a single step.

In *"Experimentation and Proof in Mathematics,"* Michael de Villiers discusses the substantial importance of experimentation for mathematical proof and its limitations. The paper rests on a wealth of historical examples and on cases from de Villiers's personal mathematical experience.

De Villiers groups his considerations around three basic subthemes: (1) the relation between conjecturing on the one hand and verification/conviction on the other; (2) the role of refutations in the process of generating a (final) proof; and (3) the interplay between experimental and deductive phases in proving.

De Villiers writes that conjecturing a mathematical theorem often originates from experimentation, numerical investigations and measurements. A prominent example is Gauss's 1792 formulation of the Prime Number Theorem, which Gauss based on a great amount of numerical data. Hence, even in the absence of a rigorous proof of the theorem, mathematicians were convinced of its truth. Only at the end of the nineteenth century was an actual proof produced that was generally accepted.

Hence, conviction often precedes proof and is, in fact, a prerequisite for seeking a proof. Experimental evidence and conviction play a fundamental role. On the other hand, this is not true in every case. Sometimes it might be more efficient to look for a direct argument in order to solve a problem rather than trying a great number of special cases.

The role of refutations in the genesis of theorems and proofs, be they global or heuristic, is a typical Lakatosian motive. De Villiers gives several examples and shows that the study of special cases and the search for counter-examples, even after a theorem has been proved, are frequently very efficient in arriving at a final, mature formulation of a theorem and its proof. Thus, this strategy belongs to the top-level methods of mathematical research and should be explicitly treated in the classroom. In this context, de Villiers argues against a radical fallibilist philosophy of mathematics by making clear that its implicit assumption that the process of proof and refutations can carry on infinitely is erroneous.

Finally, de Villiers analyses the complementary interplay between mathematical experimentation and deduction, citing several thought-provoking examples.

In "*Proof, Mathematical Problem-Solving, and Explanation in Mathematics Teaching,*" Kazuhiko Nunokawa discusses the relation between proof and exploration by analyzing concrete processes of problem solving and proof generation which he observed with students. The paper focuses on the relationships among the problem solvers' explorations, constructions of explanations and generations of understanding. These three mental activities are inseparably intertwined. Explorations facilitate understanding, but the converse is also true. Exploration is guided by understanding and previously generated (personal) explanations. Problem solvers use implicit assumptions that direct their explorative activities. They envisage prospective explanations, which in the process of exploration become real explanations (or not). An especially interesting feature of the processes of exploration and explanation is the generation of new objects of thought, a process of abstraction which eliminates nonessential conditions, leads to a generalization of the situation at hand and opens the eyes to new phenomena and theorems.

A central theme for Nunokawa is the fundamental role of diagrams and their stepwise modification in the observed problem-solving processes. Hence, at the end of his paper, Nunokawa rightly remarks that most teachers have the (bad) habit to present so-to-speak final versions of diagrams to their students, whereas it would be much more important and teachable "to investigate how the final versions can emerge through interactions between explorations and understandings and what roles the immature versions of diagrams play in that process."

Evelyne Barbin's paper "*Evolving Geometric Proofs in the 17th Century: From Icons to Symbols*" is the first of two historical case studies that conclude the volume. The wider context of her study is a reform or transformation of elementary geometry which took place in the course of the scientific revolution of the seventeenth century and might be termed "arithmetization of geometry." In the seventeenth century, a widespread anti-Euclidean movement criticized Euclid's *Elements* as aiming more at certainty than at evidence and as presenting mathematical statements not in their "natural order." Hence, some mathematicians worked on a reform of elementary geometry and tried to organize the theory in a way that not only convinced but enlightened. Two of them, Antoine Arnauld and Bernard Lamy, wrote textbooks on elementary geometry in the second half of the seventeenth century.

Barbin analyses and compares five different proofs of a certain theorem on proportions: two ancient proofs by Euclid and three proofs from the seventeenth century by Arnauld and Lamy. In the modern view, the theorem consists of a simple rule for calculating with proportions, which says that in a proportion the product of the middle members is equal to the product of the external members. In her analysis, Barbin follows a rigorous method, making one of the rare and successful attempts to apply Charles Sanders Peirce's semiotic terminology to concrete mathematical texts. Barbin explains Peirce's concepts of symbol, diagram, icon, index and representation and applies them to the different proofs. Thus, she consistently elucidates the proofs' differences and specificities of style. The result of this analysis is that the seventeenth century authors not only produced new proofs of an ancient theorem but brought about a new conception or style of proof.

In "*Proof in the Wording: Two Modalities from Ancient Chinese Algorithms*," Karine Chemla analyses the methods that early Chinese mathematicians used for proving the correctness of algorithms they had developed. The manuscripts she considers were in part recovered through excavations of tombs in the twentieth century; others have come down to us via the written tradition of Chinese mathematics. These manuscripts contain mainly algorithms; thus, it is a fundamental issue whether they contain arguments in favor of the algorithms' correctness and, if so, how these arguments are presented. Hence, in Chinese mathematics proof apparently takes a form distinctly different from the Western tradition. Nevertheless, there are certain points of similarity: Some parts of Western mathematics, for example in the seventeenth century, are presented as problems and algorithms for their solution.

In her analysis, Chemla uses a specific framework, to take into account that on the one hand most of the algorithms presuppose and refer to certain material calculating devices. Thus, it is an important question whether the algorithms present the operations step by step in regard to the calculating device. On the other hand, she has to consider in general how detailed the description of an algorithm is; thus, she writes of the "grain of the description." One of her most important results is her finding that proofs for the correctness of an algorithm are mainly given by way of semantics: that is, the Chinese authors often very carefully designated the meaning of the magnitudes calculated at each step in the course of an algorithm. In addition, the Chinese mathematicians might use a "coarser grain" of description – collapsing certain standard procedures – or change the order of operations in order to enhance the transparency of a proof.

Both these historical case studies show convincingly that proof and how it is represented strongly depend on the "diagrams" available in a certain culture and at a certain time.

In conclusion, we trust that this volume shows that much can be learned from an interdisciplinary approach bringing together perspectives from the fields of mathematics education, philosophy of mathematics, and history of mathematics. We also hope that the ideas embodied in this collection of papers will enrich the ongoing discussion about the status and function of proof in mathematics and its teaching, and will stimulate future cooperation among mathematical educators, philosophers and historians.

Acknowledgments We thank the "Deutsche Forschungsgemeinschaft" (DFG) for the generous funding of the Essen conference in 2006. We also thank the participants and authors for intense and fruitful discussions during the conference and for their generous spirit of cooperation in the preparation of this volume. A multitude of colleagues from history of mathematics, mathematics education and philosophy of mathematics refereed the various papers and thereby helped ensure the quality of this volume. We extend many thanks to them, too. We wish to thank John Holt for his stylistic polishing of much of the manuscript and for his helpful editorial advice and suggestions overall. We are also very grateful to Sarah-Jane Patterson and Stephanie Dick, graduate students in the Institute for History and Philosophy of Science and Technology at the University of Toronto, for their invaluable help in the preparation of the manuscript.

Gila Hanna, Toronto

Hans Niels Jahnke, Essen

Helmut Pulte, Bochum